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NON-LINEAR FREE VIBRATION ANALYSIS OF A STRING UNDER BENDING MOMENT EFFECTS USING THE PERTURBATION METHOD

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In this paper, the non-linear free vibration of a string with large amplitude is considered. The initial tension, lateral vibration amplitude, cross-section diameter and the modulus of elasticity of the string have main effects on its natural frequencies. Increasing the lateral vibration amplitude makes the assumption of constant initial tension invalid. Therefore, it is impossible to use the classical equation of transverse motion assuming a small amplitude. On the other hand, by increasing the string cross-sectional diameter, the bending moment effect will increase dramatically, and it will act as an impressive restoring moment. Considering the effects of the bending moments, the non-linear equation governing the large amplitude transverse vibration of a string is derived. The time-dependent portion of the governing equation has the form of the Duffing equation. Due to the complexity and non-linearity of the derived equation and the fact that there is no established exact solution method, the equation is solved using the perturbation method. The results of the analysis are shown in appropriate graphs, and the natural frequencies of the string due to the non-linear factors are compared with the natural frequencies of the linear vibration of a string without bending moment effects.

1. INTRODUCTION

The governing equation for small transverse vibrations of an elastic string which is subjected to a large tensional force was obtained more than 200 years ago, and for the first time, Kirchhoff studied non-linear vibrations of a string in 1876. Other researchers have also considered vibrations of a string which is subjected to different types of boundary and loading conditions. However, the bending moment effects have been neglected in all of the previous works. Carrier [1, 2] considered the string vibration as a combination of lateral and axial motions and obtained two equations of motion in those directions but neglected the bending moment effects.

Recently, Leissa and Saad [3] have investigated the large amplitude vibration of a string and obtained two simultaneous non-linear differential equations and solved them using the Galerkin method, neglecting the bending moment effects. Validity of the results is verified by finite difference solutions. Zhu *et al*. [4] have examined the vibrations of ballooning elastic strings. In this investigation, the non-linear dynamic response of a string which is fixed at one end and subjected to a constant circular motion at the other end has been studied. Tan and Ying [5] studied the vibrational behaviour of an axially moving string. They offered a method to obtain the longitudinal and lateral vibration response of the string. The method was derived in the frequency domain and interpreted in terms of wave propagation functions. Zhu *et al.* [6] have studied asymptotic distribution of a translating string eigenvalues using a new spectrum analysis method. The string is constrained at some arbitrary location which is modelled as a mass–spring–damper system. The asymptotic solution for eigenvalues is obtained using the characteristic equation of the coupled system of constraint and string for all parameters of the constraint. Xiong and Hutton [7] have investigated the small amplitude vibration of a string with circular motion which is restrained by point and distributed springs. The governing equation and the boundary conditions are derived using the Hamilton's principle. Terumichi *et al.* [8] have investigated the unsteady vibration of a string. The lower end of the string is attached to a mass–spring system, and the spring has a time varying length. The string is hung, and the upper end of the string is excited by a horizontal and sinusoidal displacement. Some investigations have been devoted to the measurement and control of string vibrations. Achkire *et al.* [9] have studied a non-contact measurement method for lateral vibration of a string using an analogue position sensing detector. Fung *et al.* [10] examined the control of string vibration using variable structure control (VSC) method. The objective of the investigation was suppression of the axially moving string vibrations in a short time. All of the above-mentioned studies have used either the classical or non-linear theory of string vibration neglecting the bending moment effects. The numerical methods are mainly used for vibration analysis of string which is subjected to different types of boundary and loading conditions.

Considering the bending moment effects, the non-linear and large amplitude vibration analysis of a string is studied in this paper. For the sake of brevity, the phrase of "the non-linear vibration of a wire" will be used instead of "the large amplitude and non-linear vibration of a string under bending moment effects". Here, the equation governing the large amplitude and non-linear vibration of a wire is derived and analyzed. The effects of important parameters such as an increase in vibration amplitude, or an increase in wire cross-sectional diameter and its initial tension, on natural frequencies are examined. The results obtained through the non-linear vibration analysis of a wire are compared with those of three theories: the classical theory of string vibration, the non-linear theory of string vibration, and the linear theory of wire vibration. The obtained results are shown in appropriate graphs.

2. DERIVATION OF THE EQUATION GOVERNING THE NON-LINEAR VIBRATION OF A WIRE

According to the classical theory of string vibration, the equation governing lateral vibration of the string will be the one-dimensional wave equation which has been studied thoroughly in many references, e.g., reference [11].

If the vibration amplitude of the string is large, one cannot expect that the tension force during the string vibration remains constant and in this case, the non-linear vibration of the string will be encountered. In classical and/or non-linear theory of string vibration one neglects the bending strength. On the other hand, if one attempts to examine the vibrations of a wire, the bending strength cannot be neglected. Moreover, if the large amplitude vibrations of a thick wire is considered, neither can the bending strength be neglected, nor does the tension force during the wire vibration remain constant. In such cases, the classical

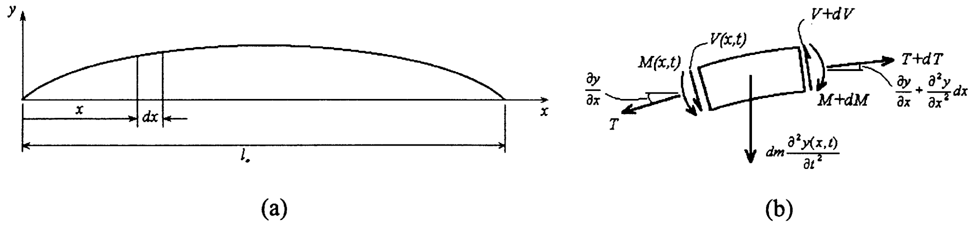

Figure 1. (a) An element of a vibrating wire. (b) The free body diagram of the element.

and the non-linear vibration of a string cannot be used. Therefore, in order to study the non-linear vibrations of a wire, one needs to derive the governing equation. For this end, consider an element of the vibrating wire of length dx, as shown in Figure 1(a) for which the free body diagram of this element is shown in Figure 1(b).

Since all of the forces are assumed to be conservative, then the Hamilton's principle may be written as [12,13]

$$\delta \int_{t_1}^{t_2} (K - P)\,\mathrm{d}t = 0, \tag{1}$$

where K is the kinetic energy and P is the potential energy of the system. Considering the transverse vibrations of the wire, the kinetic energy of the element may be expressed as follows:

$$\mathrm{d}K(t) = \frac{1}{2}\,\mathrm{d}m\left(\frac{\partial y(x,t)}{\partial t}\right)^2, \tag{2}$$

where dm is the mass of the wire element.

In general, the mass per unit length of the wire $\rho(x)$ is a function of x. Therefore, the total kinetic energy of the system at an arbitrary time t may be written as

$$K(t) = \frac{1}{2}\int_0^{l_0} \left(\frac{\partial y}{\partial t}\right)^2 \rho(x)\,\mathrm{d}x. \tag{3}$$

The potential energy of the vibrating wire may be considered to come from the three facts: (1) an initial tension of the wire, (2) an increase in length of the wire with respect to its initial length at the static equilibrium position, and (3) the presence of the bending moment effects. These three parts of the potential energy of the system may be calculated separately. As mentioned before, in the case of large amplitude vibration of the wire, the tension force cannot be assumed to remain constant, because the initial length of the wire at static equilibrium position, l_0, will increase to a large value, l, in a deflected position. Accordingly, the initial tension force T_0 will be increased to T. If the wire is made of a linear elastic material, then using the Hook's law, the tension force of the wire at any arbitrary time t may be expressed as

$$T(t) = T_0 + \frac{EA}{l_0}\,\Delta l_0(t), \tag{4}$$

where E is the modulus of elasticity, A is the cross-sectional area of the wire, and $\Delta l_0(t)$ is an increase in the initial length of the wire, l_0, at a given time t.

If $y(x, t)$ represents the deflection function at time t, the elongation of the wire at time t is

$$\Delta l_0(t) = \int_0^{l_0} \sqrt{1 + \left(\frac{\partial y(x,t)}{\partial x}\right)^2}\,\mathrm{d}x - l_0 \tag{5}$$

and the tensional force at an arbitrary time may be obtained as

$$T(t) = T_0 + \frac{EA}{l_0}\left[\int_0^{l_0} \sqrt{1 + \left(\frac{\partial y(x,t)}{\partial x}\right)^2}\,\mathrm{d}x - l_0\right]. \tag{6}$$

The initial constant tension force, T_0, causes an initial potential energy stored in the wire, P_0. According to the Hamilton's principle, equation (1), for the formulation of the system's equation of motion, the variations of the kinetic and potential energy of the system must be considered, and the variations of the initial constant potential energy, P_0, is zero, i.e., $\delta P_0 = 0$. Then P_0 may be considered as a reference level of the potential energy, and the potential energy of the system may be measured relatively with respect to that reference level.

According to the above-mentioned assumption and considering the initial tension force T_0 which remains constant, and using equation (6), the stored potential energy of the wire element with a length of dx which comes from the initial tension force, and its variations due to the variations of the wire length, can be expressed as

$$\mathrm{d}P_T(t) = \left\{T_0 + \frac{1}{2}\frac{EA}{l_0}\left[\int_0^{l_0} \sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}\,\mathrm{d}x - l_0\right]\right\}\left(\sqrt{1 + \left(\frac{\partial y}{\partial x}\right)^2}\mathrm{d}x - \mathrm{d}x\right). \tag{7}$$

Integrating the above equation along the wire length, and expanding the expressions under the radical signs gives

$$P_T(t) = \frac{1}{2}\left[T_0 + \frac{EA}{2l_0}\int_0^{l_0} \frac{1}{2}\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x\right]\int_0^{l_0}\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x. \tag{8}$$

On the other hand, the potential energy which comes from the presence of the bending moment effects is

$$P_B(t) = \frac{1}{2}\int_0^{l_0} EI(x)\left(\frac{\partial^2 y}{\partial x^2}\right)^2 \mathrm{d}x. \tag{9}$$

Substituting equations (3), (8), and (9) into equation (1), and using the variational method and applying the integration by parts gives

$$\int_{t_1}^{t_2}\left\{-\int_0^{l_0}\left[\rho(x)\frac{\partial^2 y}{\partial t^2} - \left(T_0 + \frac{EA}{2l_0}\int_0^{l_0}\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x\right)\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2}{\partial x^2}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right)\right]\delta y\,\mathrm{d}x\right.$$
$$\left. + \left[\frac{\partial}{\partial x}\left(EI(x)\frac{\partial^2 y}{\partial x^2}\right) - \left(T_0 + \frac{EA}{2l_0}\int_0^{l_0}\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x\right)\frac{\partial y}{\partial x}\right]\delta y\Big|_0^{l_0} - EI(x)\frac{\partial^2 y}{\partial x^2}\,\delta\left(\frac{\partial y}{\partial x}\right)\Big|_0^{l_0}\right\}\mathrm{d}t = 0. \tag{10}$$

Considering the arbitrariness of the virtual displacement, δy, over the domain $0 < x < l_0$, and assuming that either δy or its coefficient is zero at the boundaries, and similarly assuming that either $\delta(\partial y/\partial x)$ or its coefficient is zero at the boundary points, then the above equation can be satisfied if and only if

$$\rho(x)\frac{\partial^2 y}{\partial t^2} - \left[T_0 + \frac{EA}{2l_0}\int_0^{l_0}\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial^2 y}{\partial x^2} + \frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right] = 0, \quad 0 < x < l_0 \tag{11a}$$

and

$$\frac{\partial}{\partial x}\left[EI(x)\frac{\partial^2 y}{\partial x^2}\right]-\left[T_0+\frac{EA}{2l_0}\int_0^{l_0}\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial y}{\partial x}=0 \quad \text{at } x=0,\, x=l_0 \tag{11b}$$

or

$$y=0, \quad \text{at } x=0,\, x=l_0 \tag{11c}$$

and

$$EI(x)\frac{\partial^2 y}{\partial x^2}=0 \quad \text{at } x=0, \quad x=l_0 \tag{11d}$$

or

$$\frac{\partial y}{\partial x}=0 \quad \text{at } x=0, \quad x=l_0. \tag{11e}$$

Equation (11a) is the equation of motion governing the non-linear vibration of the wire. This equation is a non-linear fourth order partial differential equation and requires four boundary conditions. Equations (11b) and (11c) provide two boundary conditions, one at $x=0$ and the other at $x=l_0$, and the two remaining boundary conditions are provided by the equations (11d) and (11e). The choice of the four boundary conditions in equations (11b)–(11e) depends on the nature of the boundaries of the problem considered, e.g., if both the ends of the wire are hinged, equations (11c) and (11d) will be the appropriate boundary conditions. If the elastic behaviour of the wire and its cross-sectional area remain constant along its length, the equation of motion and the associated boundary conditions of the hinged wire may be written as

$$EI\frac{\partial^4 y}{\partial x^4}-\left[T_0+\frac{EA}{2l_0}\int_0^{l_0}\left(\frac{\partial y}{\partial x}\right)^2 \mathrm{d}x\right]\frac{\partial^2 y}{\partial x^2}+\rho\frac{\partial^2 y}{\partial t^2}=0 \tag{12a}$$

$$y(x,t)=0 \quad \text{at } x=0, \quad x=l_0 \tag{12b}$$

and

$$EI\frac{\partial^2 y}{\partial x^2}=0 \quad \text{at } x=0, \quad x=l_0. \tag{12c}$$

3. SOLUTION OF THE GOVERNING EQUATION

As seen from integro-differential equation (12a), the coefficient of the second term depends on the $\partial y/\partial x$ along the wire length, and the presence of this term makes this equation different from that of the beam equation of motion. In order to examine the influence of the parameters, such as the wire diameter, vibration amplitude of the wire, and its initial elongation, on the natural frequencies corresponding to the individual natural modes of the non-linear vibration of the wire, the system response at the rth vibrational mode is proposed as follows:

$$y_r(x,t)=G_r(t)c_r\sin\beta_r x, \tag{13}$$

where c_r and β_r are constants.

One advantage of considering the solution of the governing equation at any vibrational modes individually as in equation (13) is that the governing equation will be separable and the non-linear vibration of the wire can be easily examined in an arbitrary vibrational mode. However, if the solution of the equation is considered as a general form of the combination of contribution of all the natural modes, the governing equation cannot be separated, and the problem becomes complicated.

By substituting equation (13) into equation (12), the governing equation may be separated. Then, the spatial dependent part of the solution, i.e., $Y_r(x) = c_r \sin\beta_r x$, must satisfy the boundary conditions. It is obvious that the boundary conditions at $x = 0$ are satisfied automatically, and for the boundary conditions at $x = l_0$

$$Y_r(x)|_{x=l_0} = EI\left.\frac{\mathrm{d}^2 Y_r(x)}{\mathrm{d}x^2}\right|_{x=l_0} = 0 \Rightarrow c_r \sin\beta_r l_0 = -EIc_r\beta_r^2 \sin\beta_r l_0 = 0. \tag{14}$$

The above equation has an infinite number of solutions as

$$\beta_r = \frac{r\pi}{l_0}, \quad r = 1, 2, \ldots. \tag{15}$$

Moreover, since both the ends of the wire are simply supported, then relation (13) will be the solution of equation (12a) at the rth mode. In fact, due to this difference in the form of the boundary conditions of the vibrating wire and those of the vibrating string, the bending moment effects are highlighted. Substituting equation (13) into equation (12a) gives

$$\begin{aligned} &EIc_r G_r(t)\beta_r^4 \sin\beta_r x - \left[T_0 + \frac{EA}{2l_0}\left(\frac{1}{2}G_r^2(t)c_r^2\beta_r^2 l_0\right)\right] \\ &[-\beta_r^2 c_r G_r(t)\sin\beta_r x] + \rho c_r \sin\beta_r x\ddot{G}_r(t) = 0. \end{aligned} \tag{16}$$

Dividing the above equation by $c_r \sin\beta_r x$, and rearranging the obtained equation gives

$$\ddot{G}_r(t) + \frac{EAc_r^2\beta_r^4}{4\rho}G_r^3(t) + \beta_r^2\left(\frac{T_0 + EI\beta_r^2}{\rho}\right)G_r(t) = 0. \tag{17}$$

Equation (17) is the time-dependent part of (12a), and is a non-linear equation.

The perturbation method is one of the most appropriate methods which can be used to solve this type of equation. It provides a useful approximate analytical tool for solving a large class of non-linear equations. Using the perturbation method, a complex non-linear problem may be decomposed into an infinite number of relatively easy ones. In this method, the solution of the perturbated problem is considered as the sum of the solution of the unperturbated problem and a sequence of terms/functions with increasing power of a small parameter as their coefficients, so that the first few terms reveal the important feature of the solution.

Unlike the numerical methods, the perturbation approach provides a good physical insight into the problem considered, and the influence of all the parameters on the solution can be assessed easily.

Due to the large coefficient of the non-linear term in equation (17), the perturbation method cannot readily be applied in order to solve the equation. For this purpose, a new function is introduced, $g_r(t)$, as

$$\frac{T_0 + EI\beta_r^2}{\rho}G_r(t) = g_r(t). \tag{18}$$

Substituting equation (18) into equation (17), and arranging terms gives

$$\ddot{g}_r(t) + \frac{EA\rho c_r^2 \beta_r^4}{4(T_0 + EI\beta_r^2)^2} g_r^3(t) + \beta_r^2 \left(\frac{T_0 + EI\beta_r^2}{\rho}\right) g_r(t) = 0, \quad r = 1, 2, \ldots . \tag{19}$$

The above equation has a solution for a given r, and a sequence of solutions corresponding to $r = 1, 2, 3, \ldots$ is obtained. Equation (19) may be written in a more simple form if the coefficients of the terms $g_r(t)$ and $g_r^3(t)$ are represented by the new parameters p_r and ε_r, respectively, as

$$\ddot{g}_r(t) + \varepsilon_r g_r^3(t) + p_r g_r(t) = 0. \tag{20}$$

If the cross-section of the wire is a circle with diameter D, mass per unit volume ρ_v, and denoting the difference between the stretched and unstretched length of the wire by Δ, the coefficients, p_r and ε_r, in equation (20) may be obtained as

$$p_r = \frac{Er^2\pi^2}{\rho_v l_0^4}\left[l_0\Delta + \left(\frac{r\pi D}{4}\right)^2\right], \qquad \varepsilon_r = \frac{\rho_v c_r^2 (r\pi)^4}{4l_0^4 E[\Delta/l_0 + (r\pi D/4l_0)^2]^2}. \tag{21a, b}$$

It is obvious from equation (20) that by omitting the non-linear term, p_r will be equal to the square of the natural frequency of the rth linear vibrational mode of a wire ω_{0r}:

$$p_r = \omega_{0r}^2. \tag{22}$$

Equation (20) has a special form which is known as the *Duffing* equation, and it may be solved using the techniques introduced in references [14, 15]. If the wire is deformed to a sine wave shape with an amplitude of c_r, i.e., $Y_r(x) = c_r \sin \beta_r x$, and released from the rest at time $t = 0$, then

$$y_r(x, t) = G_r(t) c_r \sin \beta_r x,$$

$$y_r(x, t)|_{t=0} = c_r \sin \beta_r x \Rightarrow G_r(0) = 1, \qquad \dot{y}_r(x, t)|_{t=0} = 0 \Rightarrow \dot{G}_r(0) = 0. \tag{23a, b}$$

Considering equation (18) and the above initial conditions gives

$$g_r(0) = (T_0 + EI\beta_r^2)/\rho, \qquad \dot{g}_r(0) = 0. \tag{24a, b}$$

In equation (20), p_r and ε_r are positive constants; moreover, ε_r is the coefficient of the non-linear term and it is a small parameter which depends on the initial tension, the mechanical properties and the geometrical dimensions of the wire as well as the mode number and the corresponding amplitude of the natural vibration mode. Therefore, $g_r(t)$ at each vibration mode, and its corresponding natural frequency of the non-linear vibration, ω_r, can be expressed as the following sequences with increasing powers of ε_r:

$$g_r(t) = g_{0r}(t) + \varepsilon_r g_{1r}(t) + \varepsilon_r^2 g_{2r}(t) + \cdots, \qquad \omega_r^2 = \omega_{0r}^2 + \varepsilon_r \alpha_1 + \varepsilon_r^2 \alpha_2 + \cdots, \tag{25, 26}$$

where $g_{ir}(t)$ and α_i are unknown.

Substituting $g_r(t)$ and ω_r^2 into equation (20) gives

$$\ddot{g}_{0r}(t) + \varepsilon_r \ddot{g}_{1r}(t) + \varepsilon_r(g_{0r}^3 + 3\varepsilon_r g_{0r}^2 g_{1r} + \cdots) + (\omega_r^2 - \varepsilon_r \alpha_1)(g_{0r} + \varepsilon_r g_{1r}) = 0. \tag{27}$$

The above equation implies that the coefficients of any powers of ε_r have to be equal to zero, then

$$\ddot{g}_{0r} + \omega_r^2 g_{0r} = 0, \qquad \ddot{g}_{1r}(t) + \omega_r^2 g_{1r} = \alpha_1 g_{0r} - g_{0r}^3. \qquad \vdots \tag{28a, b}$$

Applying the initial conditions, and solving the above equations gives

$$g_{0r}(t) = \frac{T_0 + EI\beta_r^2}{\rho}\cos\omega_r t, \tag{29}$$

$$g_{1r}(t) = \frac{(T_0 + EI\beta_r^2)^3}{32\rho^3\omega_r^2}(\cos 3\omega_r t - \cos\omega_r t), \qquad \alpha_1 = \frac{3}{4}\left(\frac{T_0 + EI\beta_r^2}{\rho}\right)^2. \tag{30, 31}$$

Upon substituting equations (29) and (30) into equation (25), and changing the variables in equation (18), the following results are obtained:

$$g_r(t) = \frac{T_0 + EI\beta_r^2}{\rho}\cos\omega_r t + \frac{\varepsilon_r(T_0 + EI\beta_r^2)^3}{32\rho^3\omega_r^2}(\cos 3\omega_r t - \cos\omega_r t), \tag{32a}$$

$$G_r(t) = \cos\omega_r t + \frac{\varepsilon_r p_r^2 l_0^4}{32\omega_r^2(r\pi)^4}(\cos 3\omega_r t - \cos\omega_r t), \tag{32b}$$

then the solution $y_r(x, t)$ will be

$$y_r(x, t) = c_r \sin\frac{r\pi x}{l_0}\left\{\cos\omega_r t + \frac{\varepsilon_r p_r^2 l_0^4}{32\omega_r^2(r\pi)^4}(\cos 3\omega_r t - \cos\omega_r t)\right\}. \tag{33}$$

Equation (33) represents the response of the stretched simply supported wire to a sine wave shape initial deflection with an amplitude of c_r. This equation shows that if the wire is initially deformed into one of its natural mode shapes, the response will occur purely at the excited mode. In this case, it is easy to examine the influence of the parameters like wire diameter, initial elongation of the wire and its vibration amplitude on any natural frequency of the desired vibrational mode. On the other hand, given the geometrical dimensions and the mechanical properties of the wire as well as its initial tension and vibration amplitude, the non-linear behaviour of the wire at different natural modes can be investigated. Moreover, having calculated the response of the system to a sine wave shape initial deflection at the rth mode (equation (33)), the response of the system to an arbitrary initial deflection function can be obtained by combining the responses $y_r(x, t)$ of all modes and obtaining the unknowns, c_r's. Here, as mentioned, the non-linear behaviour of individual modes and their corresponding natural frequencies is considered.

Substituting equation (31) into equation (26), the angular natural frequencies of the non-linear vibration of the wire may be calculated as

$$\omega_r^2 = \omega_{0r}^2 + \frac{3\varepsilon_r}{4}\left(\frac{T_0 + EI\beta_r^2}{\rho}\right)^2,$$

$$= p_r + \frac{3\varepsilon_r p_r^2 l_0^4}{4(r\pi)^4}. \tag{34}$$

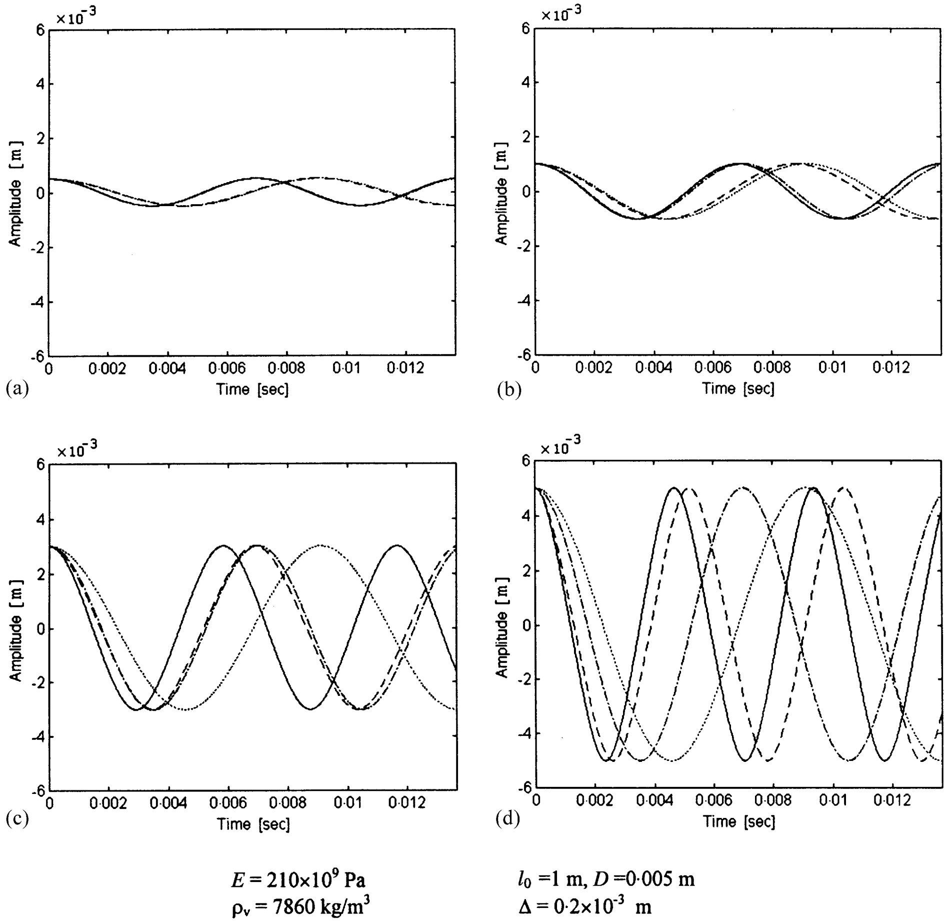

Figure 2. The influence of vibration amplitude on the results obtained using the four theories in the third mode: (a) $c_3 = 0{\cdot}5 \times 10^{-3}$ m, (b) $c_3 = 0{\cdot}001$ m, (c) $c_3 = 0{\cdot}003$ m, and (d) $c_3 = 0{\cdot}005$ m. ········, linear vibration of a string; -.-.-.-, linear vibration of a wire (bending moment effects); -------- non-linear vibration of a string; ——, non-linear vibration of a wire (the effects of the bending moments and the vibration amplitude).

Therefore, the rth natural frequency of the non-linear vibration of the wire in Hertz will be

$$f_r = \frac{1}{2\pi}\sqrt{p_r + \frac{3\varepsilon_r p_r^2 l_0^4}{4(r\pi)^4}}, \tag{35}$$

where ε_r and p_r are defined in equations (21a) and (21b).

The accuracy of the solution provided by the perturbation method depends on a number of facts such as the order of magnitude of the chosen small parameter ε_r, the order of problem non-linearity, and the number of terms which are considered in the perturbation expansion.

If the magnitude of the non-linear term in the equation is very small compared with those of the other terms, a few first terms of the expansion will provide an appropriate accuracy of the solution and in this case, the remaining terms of the expansion will have a small contribution to the solution. Therefore, one may neglect those terms. In the numerical examples considered in the next section non-linear vibration of a wire is investigated. In these examples, ε_r is much smaller than unity, e.g., for the cases plotted in Figures 2 and 3,

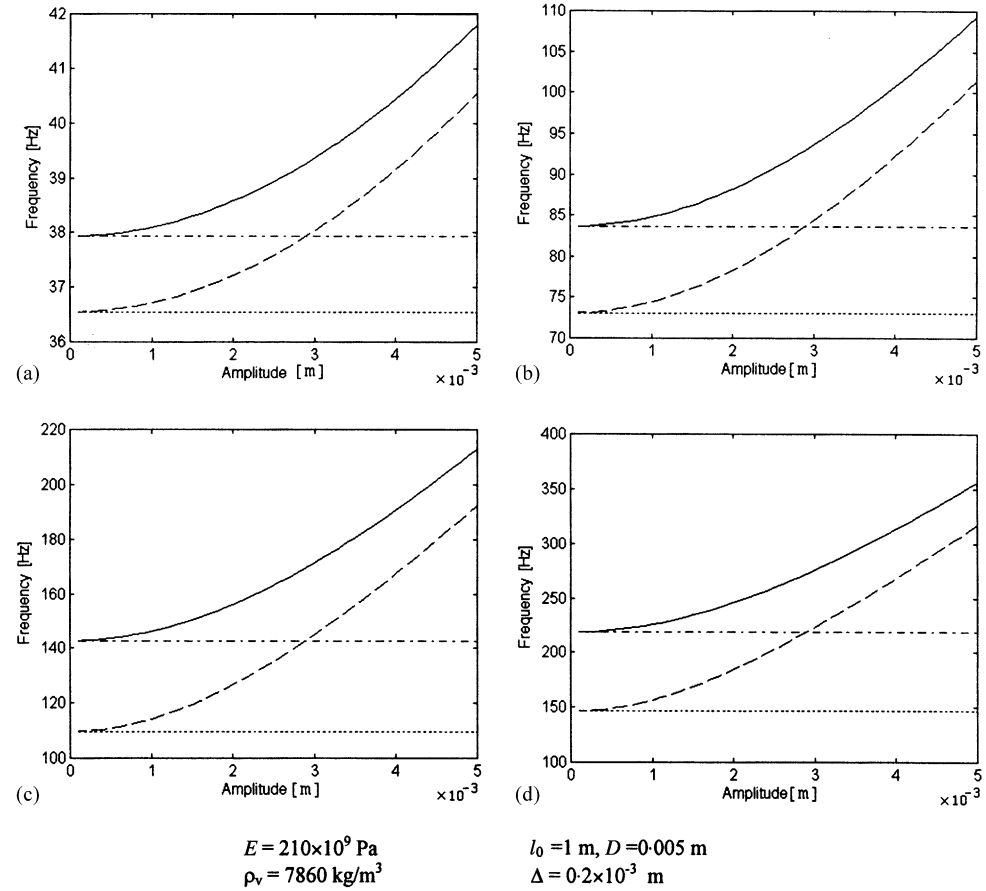

Figure 3. The influence of the vibration amplitude on the first four natural frequencies of the vibrating string and the wire according to the four theories: (a) the first mode, (b) the second mode, (c) the third mode, and (d) the fourth mode. ········, linear vibration of a string; -.-.-.-, linear vibration of a wire (bending moment effects); - - - - - - - non-linear vibration of a string; ——, non-linear vibration of a wire (the effects of the bending moments and the vibration amplitude).

the minimum value of ε_r is $1{\cdot}6 \times 10^{-4}$ and $4{\cdot}9 \times 10^{-4}$, respectively, and the maximum value of ε_r for the mentioned cases is 0·016 and 0·029, respectively. In such cases, the solution obtained by neglecting the terms with higher powers of ε_r, i.e., 2, 3, …, will provide excellent accuracy for the approximate solution. The solution will present the non-linear behaviour of the wire adequately. However, in order to study very large amplitude vibrations of the wire, one must consider more terms of the perturbation expansion series.

4. ILLUSTRATIVE EXAMPLES

In this section, the large amplitude and non-linear free vibration behaviour of a stretched wire with the given geometrical dimensions and mechanical properties is examined, and the results are compared with those of the linear and non-linear vibration of a string as well as with those of the linear vibration of a wire.

Equations (33) and (35) represent the large amplitude vibrational motion of a wire and its rth mode natural frequency. These two equations have been derived by considering the effects of the bending moments and an increase in vibration amplitude. Due to these effects, the equation governing the non-linear vibration of the wire has a non-linear behaviour. By neglecting the appropriate terms of the above-mentioned equations, one may achieve the results of the classical theory of the string vibration, the non-linear theory of the string vibration, as well as the linear theory of the wire vibration. If the coefficient of the non-linear term, i.e., ε_r, is equal to zero, then the linear vibrational behaviour of the wire will be obtained, and if the bending moment effects are neglected i.e., $EI \simeq 0$, the non-linear vibrational behaviour of a string will be resulted. Moreover, if both of the above-mentioned effects are disregarded, then the results match those of the classical theory of the string vibration.

In order to examine the influence of an increase in vibration amplitude on the rate of non-linear behaviour of the wire vibration in the third mode, the motion of a wire particle is plotted in Figure 2 against time, and the results are compared with those of the three theories, i.e., the classical theory of a string vibration, the linear vibration of a wire, and the non-linear vibration of a string. It is assumed that the wire is made of the steel with $\rho_v = 7860\ \mathrm{kg/m^3}$, $E = 210 \times 10^9$ Pa, diameter $D = 0{\cdot}005$ m, the length at the static

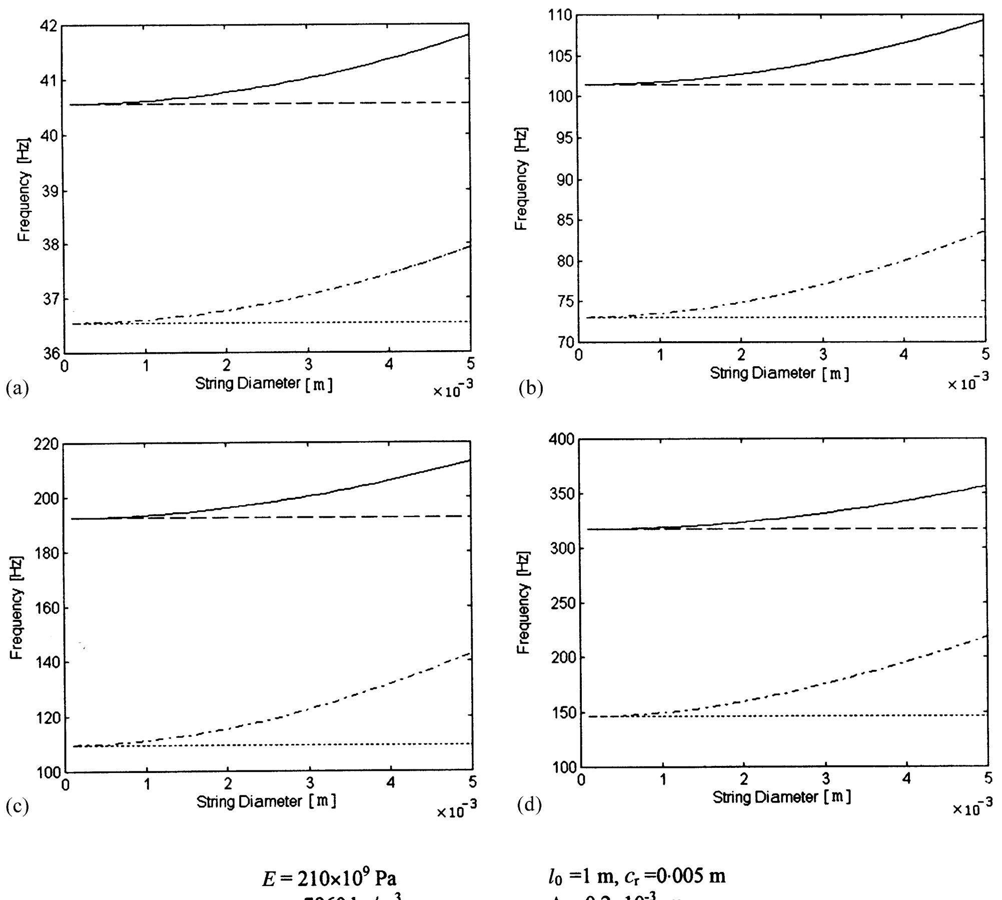

Figure 4. The effect of increasing diameter of the string on its natural frequencies according to the four theories: (a) the first mode, (b) the second mode, (c) the third mode, and (d) the fourth mode. ········, linear vibration of a string; -.-.-.-, linear vibration of a wire (bending moment effects); -------- non-linear vibration of a string; ——, non-linear vibration of a wire (the effects of the bending moments and the vibration amplitude).

equilibrium position $l_0 = 1$ m, and with the elongation (the difference between the stretched and unstretched lengths) $\Delta = 0{\cdot}2 \times 10^{-3}$ m.

It is obvious from Figure 2(a) that non-linear effects for the small amplitude vibration, i.e., $c_3 = 0{\cdot}5 \times 10^{-3}$ m, of the wire are very small. For such a case, the results of the two theories for linear and non-linear vibrations of a string are the same, and also the linear and the non-linear vibration theory of the wire give the same result. However, by increasing the vibration amplitude, the four above-mentioned theories show a quite different behaviour Figure 2(d) shows the great difference between the four theories when the large amplitude vibration of a string or wire is considered.

Figure 3 shows the influence of the diameter on the first four natural frequencies of the vibrating string and the wire, based on the above four theories. It is seen that by increasing the string diameter, the natural frequencies of all vibration modes obtained by application of two theories, i.e., the classical theory of the linear vibration of the string and the linear vibration theory of the wire, remain constant. However, the natural frequencies

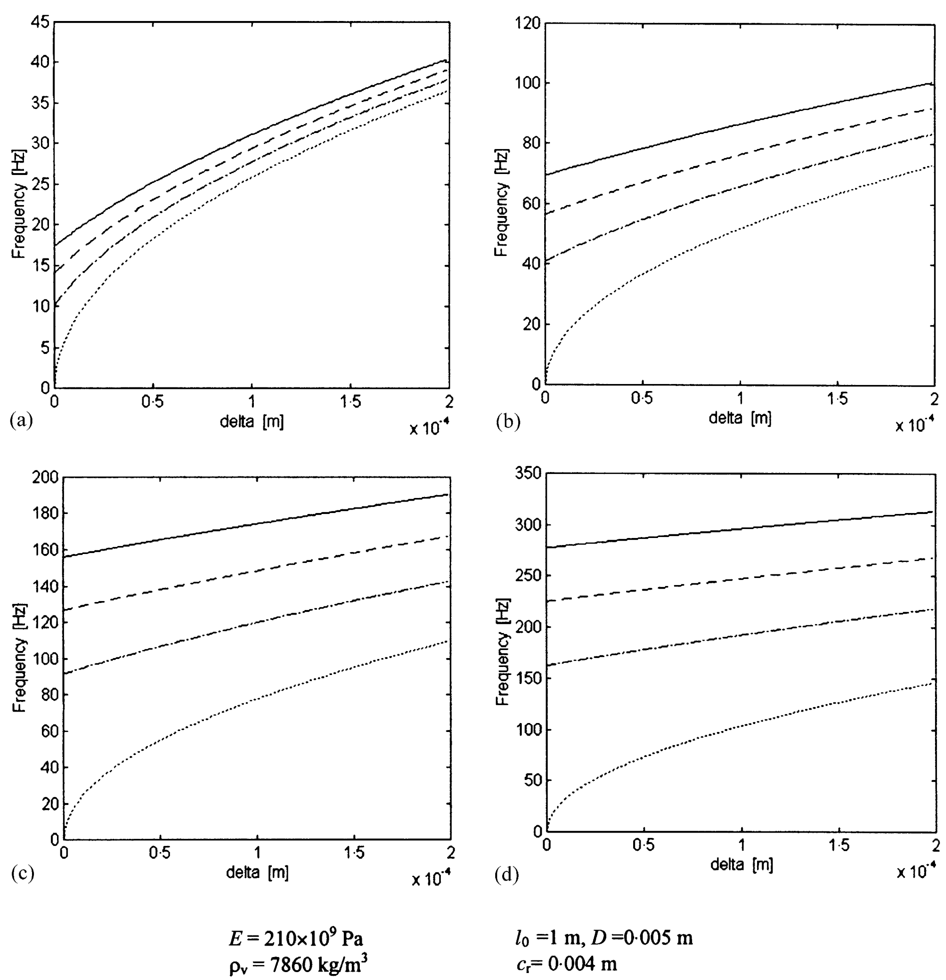

Figure 5. The effect of initial elongation at the static equilibrium position of the string and the wire on the natural frequencies according to the four theories: (a) the first mode, (b) the second mode, (c) the third mode, and (d) the fourth mode. ·······, linear vibration of a string; -.-.-.-, linear vibration of a wire (bending moment effects); ------- non-linear vibration of a string; ——, non-linear vibration of a wire (the effects of the bending moments and the vibration amplitude).

corresponding to both the theories of the non-linear vibration of the string and the wire are increased. The difference among the natural frequencies with a given vibration amplitude obtained using the four theories are mainly due to the effect of bending moments and the non-linear effect of vibration amplitude increase.

Figure 4 shows the influence of increase in diameter of the string on its natural frequencies at different modes with a large amplitude vibration, i.e., $c_r = 0{\cdot}005$ m, according to the four theories. When the diameter is small, the classical theory of the string vibration coincides with the linear vibration theory of the wire, and the non-linear vibration theory of the string coincides with the non-linear vibration theory of the wire for all vibrational modes. This shows that if the diameter of the wire is very small, the bending moment effects will be negligible in the vibrational behaviour of the wire. However, if the wire has a large diameter, then the bending moment effects can not be ignored.

The influence of an increase in initial elongation, Δ, on the natural frequencies of the string and the wire at different modes with a vibration amplitude of $c_r = 0{\cdot}004$ m, is shown in Figure 5. This figure shows that under the constant vibration amplitude, by increasing the mode number, r, the difference among the natural frequencies which are obtained using the different theories are dramatically increased. The trend of the variations of the natural frequencies in this figure shows that the bending moment and additional tension affect the natural frequencies at higher modes tremendously. In other words, the effects of excessive tension of the wire during the lateral motion of the wire, causing the non-linear behaviour of the vibration as well as the bending moment effects, will increase dramatically at higher modes. As we know, based on the classical theory of the string vibration, the string could not vibrate without the initial tension, i.e., ignoring the gravity force, the natural frequency of an unstretched string, fixed at two ends, will be equal to zero. However, according to Figure 5, and based on the three remaining theories, the natural frequencies of an unstretched string or wire is not zero, and this is an important result which coincides with the physical facts. In fact, when an elastic wire is fixed between the two points, any deflection can introduce tensional force and bending moments acting as the restoring forces, and supply the system with the potential energy required to vibrate.

5. CONCLUSION

As it is seen from the previous sections of the paper, the bending moment is the most important factor which changes the vibrational behaviour of the wire and the string. Unlike the string, the bending moment acts as an additional restoring force in wire which increases the capability of storing the potential energy. If the wire diameter and its vibration amplitude are small, one may use the classical theory of the string vibration to predict the vibrational behaviour of the wire with an acceptable accuracy, but by increasing the diameter of the wire, there will be a great difference between the results obtained through the application of non-linear theory of the wire vibration and those of the classical theory of string vibration. For a given vibration amplitude, the difference will increase tremendously by increasing the mode number (Figure 4).

Another phenomenon which influences the natural frequencies of the vibrating wire is the value of vibration amplitude. By increasing the vibration amplitude, the coefficient of the non-linear term in the governing equation, ε_r, will increase. This parameter has a dominant effect on natural frequencies, and by increasing ε_r, the natural frequencies of the wire increase monotonically, e.g., as shown in Figure 3(a), the first natural frequency of a steel wire with a length of 1 m, and a diameter of $0{\cdot}005$ m, with an amplitude of $0{\cdot}5 \times 10^{-3}$ m, which is stretched by $0{\cdot}2 \times 10^{-3}$ m initially, is approximately 38 Hz, and it almost coincides with the result obtained through the linear theory of the wire vibration. However, its natural frequency at the same mode and with an amplitude of 0·0045 m increases to 41 Hz. Therefore, a 0·004 m increase in vibration amplitude of the mentioned wire at the first mode causes 3 Hz increase in the first natural frequency. Now consider the influence of an increase in vibration amplitude on natural frequencies of the wire at the fourth mode. It is obvious from Figure 3(d) that the fourth natural frequency of the wire with the above-mentioned characteristics and with a vibration amplitude of $0{\cdot}5 \times 10^{-3}$ m is about 222 Hz. However, the natural frequency of the wire at the same mode, and with an amplitude of 0·0045 m is increased almost to 335 Hz. By comparing these two cases, it is seen that the natural frequencies of the wire at a given mode is highly related to the vibration amplitude. Therefore, one may conclude that the wire diameter and vibration amplitude are the two parameters which have predominant effects on vibrational behaviour, especially on the natural frequencies, these effects cannot be neglected.

The third parameter which affects the natural frequencies is the initial tension force of the wire. As Figure 5 shows, by increasing the initial elongation, Δ, the natural frequencies of all vibrational modes which are obtained using the four different theories are increased, and the difference between the four theories at higher initial tensions is diminished. In order to avoid the plastic deformation, it should be noted that the total strain at any point of the vibrating wire must be smaller than the yield strain of the wire material. The total strain which occurs in the wire material is mainly composed of three parts: the strain caused by the initial tension, the strain caused during the vibration by the additional elongation of the deformed wire, and the strain caused by the bending moments.

Reviewing all the figures, it is seen that the classical theory of the string vibration and the non-linear vibration theory of the wire are the lower and the upper limits of the four mentioned theories, respectively, and in some extreme cases, the difference between the two theories for predicting the natural frequencies is more than 100% (Figures 3(d), 4(d), and 5(d)). Therefore, if one uses the classical theory of the string vibration, the non-linear theory of string vibration, or the linear vibration theory of the wire in order to approximate the natural frequencies of the large amplitude vibration of the wire, the results will be inaccurate and the use of these theories is not recommended.

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APPENDIX A: NOMENCLATURE

A	the cross-sectional area of the string
c_r	the vibration amplitude of the rth natural mode of the wire
D	the string diameter
$\mathrm{d}m$	the mass of the wire element
E	the modulus of elasticity
f_r	the rth natural frequency of the non-linear vibration of the wire
$G_r(t)$	the time-dependent portion of the solution corresponding to the rth natural mode
$g_r(t)$	a new time-dependent function defined as $g_r(t) = [(T_0 + EI\beta_r^2)/\rho]\,G_r(t)$
I	the cross-sectional area moment of inertia about an axis normal to x and y and passing through the centre of the cross-sectional area of the wire
K	the kinetic energy
l	the length of the vibrating string at an arbitrary time
l_0	the initial length of the string at the static equilibrium position
$M(x, t)$	the bending moment
p_r	$= Er^2\pi^2/\rho_v l_0^4[l_0\Delta + (r\pi D/4)^2]$
P	the potential energy
P_0	the initial potential energy stored in the wire which comes from the initial tension force, T_0
$P_B(t)$	the potential energy which is caused by the bending moments
$P_T(t)$	the potential energy which is caused by the tension force
t	time
T	the tension force at a given point of the vibrating string at an arbitrary time
T_0	the initial tension force of the string at the static equilibrium position
$V(x, t)$	the shear force
x	the spatial co-ordinate
$y(x, t)$	the deflection of the wire at an arbitrary point x and time t
$Y_r(x)$	the spatial-dependent part of the solution which represents the rth natural mode shape
β_r	a frequency-dependent parameter
Δ	the initial elongation of the wire
$\Delta l_0(t)$	the difference between the vibrating wire length at time t and the length of the stretched wire at the static equilibrium position
ε_r	$= \rho_v c_r^2(r\pi)^4/4l_0^4 E[\Delta/l_0 + (r\pi D/4l_0)^2]^2$, a small parameter which depends on the initial tension, the mechanical properties and the geometrical dimensions of the wire as well as the mode number and the corresponding amplitude of the natural vibration mode which introduces the non-linearity into the equation
ρ	the mass per unit length of the wire
ρ_v	the mass per unit volume of the wire
ω_r	the rth circular natural frequency of the non-linear vibration of the wire
ω_{0r}	the rth circular natural frequency of the linear vibration of the wire